

Teaching and learning discrete mathematics

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Abstract

In this paper, we provide an overall perspective on the teaching and learning of discrete mathematics. Our aim is to highlight what research has been conducted in this area and to connect it to existing research ideas for future work. We begin by characterizing discrete mathematics and its role in the school curriculum, highlighting themes, topics, and mathematical practices that distinguish discrete mathematics. We then present potential benefits of focusing on discrete mathematics topics for mathematics education; in particular, we discuss the accessibility of topics in discrete mathematics, the connection to mathematical processes and affect, and the relevance of discrete mathematics in our current society. We also emphasize discrete mathematics from an international perspective, highlighting studies from the US, Italy, France, Chile, and Germany, which are across all school levels—primary, middle, and secondary school, and with some implications for post-secondary education. We particularly discuss discrete topics including number theory, combinatorics, iteration and recursion, graph theory, and discrete games and puzzles; we describe and situate these topics within literature. We also suggest the additional topics of game theory and the mathematics of fairness that we hope to see addressed in future studies.

1 Introduction

Our intention is that this paper provide an overall perspective on the teaching and learning of discrete mathematics by discussing some past and current research on the teaching and learning of discrete mathematics. The intent is to promote further research, particularly in some under-researched areas. In particular, this paper

- describes discrete mathematics and its role in the school curriculum,
- discusses some general possible affordances for learning discrete mathematics,
- describes these affordances in more detail with respect to a variety of discrete topics,
- examines the current state of research on the teaching and learning of discrete mathematics in schools worldwide, and
- suggests areas where research is lacking and proposes questions for further exploration.

We note that this special issue supports these goals by including studies and analyses from the US, Italy, France, Chile, and Germany, across all school levels—primary, middle, and secondary school, and with some implications for post-secondary education.

2 Characterizing discrete mathematics for mathematics education

To begin, we must tackle the obvious yet difficult question: What is discrete mathematics? Most broadly, the term “discrete” is in contrast to the term “continuous,” and thus discrete mathematics can be viewed as mathematics that deals with discrete rather than continuous mathematical structures and objects. In short, we characterize discrete mathematics as involving topics related to discrete objects and structures (such as integers in a domain like combinatorics), as opposed to topics focusing on continuous objects (such as real numbers in a domain like calculus). A more precise

definition is much debated and difficult to articulate (Maurer, 1997).

In our attempt to characterize discrete mathematics, we focus on discrete topics that research indicates are appropriate for school level mathematics. For example, some research describes how some of these topics can be integrated into the current curriculum in ways that both enhance and support the current curriculum (Amit & Neria, 2008; Blanton & Kaput, 2005; Carraher et al., 2008; Kenney, 1996; Radford, 2008; Rivera & Becker, 2008; Steele, 2008; Yeap & Kaur, 2008). Other research indicates how discrete topics support the competencies (such as problem solving, argumentation, communicating, and modeling) that have become increasingly more important in mathematics education (Batanero et al., 1997; Dawkins & Roh, 2022; Doorman & Gravemeijer, 2009; Greefrath & Vorhölter, 2016; Grenier & Payan, 1999; Guerrero-Ortiz et al., 2018; Lockwood et al., 2021; Maher et al., 2011; Street & Street, 1998; Vogel, 2005; Weigand, 2014).

The discrete topics discussed in papers included in this special issue are:

- o Number theory
- o Combinatorics
- o Iteration, recursion, sequences, difference equations
- o Vertex-edge graphs
- o Discrete puzzles and games

Other discrete topics that we consider relevant are:

- o Game theory
- o Voting
- o Fair division
- o Cryptography
- o Information processing

These topics cannot be taken in isolation of what are considered core mathematical practices, processes, and competencies. Just as there are disagreements about the different topics that comprise discrete mathematics, there are different lists of mathematical competencies. As reported in Niss et al. (2016):

- Australia issued a report in 1994 describing processes including investigating, conjecturing, problem solving, using mathematical language.
- In the late 1990s and early 2000s, Denmark issued reports in which they listed eight mathematical competencies: mathematical thinking, problem handling, modeling, reasoning, representations, symbols and formalism, communication, and tools. Their report influenced a number of other countries, for example Germany.

- By 2011, Korea had added two more core competencies, creativity and information processing.
- Latin American countries rephrased these processes as capabilities, competencies or abilities.
- In the United States, the National Council of Teachers of Mathematics through its “Process Standards”, recommended the competencies of reasoning, problem solving, communication, making connections, and use of representations (NCTM, 2000). These are basically supported by the Common Core State Standards Initiative “Standards for Mathematical Practice” (NGA and CCSSO, 2009).

It is precisely in the development of such process-related competencies that discrete mathematics and discrete models are accorded special importance at school (Cai, 2010; Doorman & Gravemeijer, 2009; Goldin, 2010, 2018). Therefore, it is also useful to consider research on the development of such competencies through activities in discrete mathematics contexts.

In addition, there is another important area, affect, which influences the learning of mathematics and can also play a special role in the context of learning discrete mathematics (Gibson, 2012; Goldin, 2018; Hart & Martin, 2018). We address this potential through the teaching of discrete mathematical topics in more detail in Sect. 4.2.

Based on the combined experience of the authors of this paper (along with other teachers, researchers and curriculum developers), and acknowledging the diverse range of discrete mathematics topics and related mathematical competencies, we propose a focused, practical characterization of discrete mathematics, aimed at clarifying discrete mathematics for the school curricula (Anderson et al., 2004; DeBellis & Rosenstein, 2004; Dolgos, 1990; Ferrarello & Mammanna, 2018; Hart, 1985, 1997; Hart et al., 2008; Hart, 2010; Hart & Martin, 2018; Hirsch et al., 2015; Kenney, 1996; Kenney & Hirsch, 1991; Lockwood et al., 2020a, 2020b; NCTM, 1989, 2000; Ouvrier-Buffet, 2020; Rosenstein, 2007; Sandefur, 1997; Sandefur et al., 2018). This leads to the following curricular characterization of discrete mathematics in the schools, in terms of key themes, topics, and distinctive mathematical competencies:

- *Key themes* that can be addressed through discrete mathematics: networks, enumeration, sequential change, strategic decision making, fairness, and the Internet.
- Specific discrete *topics* that address these themes: graph theory, combinatorics, iteration and recursion, game theory, the mathematics of voting and fair division, discrete games and puzzles, and number theory including some mathematics of information processing such as coding and cryptography.

- *Distinctive mathematical practices* emphasized in the teaching and learning of discrete mathematics: recursive thinking, combinatorial reasoning, algorithmic problem solving, discrete optimization, and, above all, discrete mathematical modeling.

We believe discrete mathematics can be a key component for many countries in helping their students achieve the mathematical power and competencies they desire. In later sections of this paper, we will elaborate on the identified discrete topics and themes, along with their interplay with key mathematical practices, as we discuss past, current, and proposed future state of the art research on teaching and learning discrete mathematics.

3 Discrete mathematics in school curricula

As a result of the Cockcroft Report (Cockcroft, 1982) in England and the NCTM Standards (NCTM, 1989) in the US, there was an effort to reform the teaching of mathematics. Similar discussions took place in other countries (Niss et al., 2016). These discussions promoted competencies such as problem-solving, reasoning, discussion, group work, and use of digital tools.

Much of the reform movement was to ensure the education of all students, and was seen as a way to bring equity into the teaching of mathematics. This was expressed best by the NCTM's call for 'Mathematics for All' (Schoenfeld, 2004; Wright, 2012). As such, the reforms were as much about the processes and competencies developed in studying mathematics as they were the particular topics themselves. It was argued that a number of discrete topics might be more accessible and attractive to students having difficulty with the standard curriculum (NCTM, 1989).

In addition, because of the increasing prevalence of technology in the workplace, many of these discrete topics were becoming important in different applications of mathematics in a variety of fields, and were seen by some as essential to students' mathematical literacy.

However, this reform movement did not always go forward. There was a backlash by the traditionalists in the US, for example, because it was claimed that these changes led to a less rigorous mathematics curriculum. A revision of the Standards (NCTM, 2000) aimed at a compromise of a balance between skills and process. This revision deemphasized many of the discrete topics promoted in the original Standards. These political battles over the mathematics curriculum resulted in discrete mathematics largely being ignored in these countries. Of the discrete mathematics topics specified above, very few are part of the standard curriculum in

most of the countries we are familiar with. Through personal communication, it appears that:

- Combinatorics is included in the secondary curriculum of several countries, including Spain, US, England, Germany, Hungary, Brazil, Israel.
- Connecting recursive patterns and sequences with algebraic formulas is taught, to some degree, in Spain, Germany, and the United States.
- Graph theory is included in Italy and some isolated state curriculum in the US.

In England, students focusing on mathematics can take a special track in which they have extensive exposure to discrete mathematics. In many countries, it appears that the opportunities for dealing with discrete mathematics in schools, especially when it goes beyond combinatorics, are often only seen on the level of optional recreational mathematics (Colipan & Liendo, 2022; Gravier & Ouvrier-Buffer, 2022; Greefrath et al., 2022).

We wonder if one contributing issue for the lack of discrete topics taught in the schools may be that the term 'discrete mathematics' is not well understood. Perhaps, each of the discrete topics mentioned above should be considered individually. For example, fair division algorithms and economic game theory are almost self-explanatory. While combinatorics sounds complicated, counting is clearly important. Instead of using the term 'graph theory' which may be misleading to some, we could talk about vertex-edge graphs or networks, which most people are familiar with. Iteration can be described in terms of simple recursive situations, such as repeatedly folding a piece of paper or compounding of interest, along with its accessibility through the use of spreadsheets. Therefore, it might be more productive to talk about the individual discrete topics than the discipline as a whole. This is also intended to describe the mentioned topic areas of discrete mathematics for school more clearly once again. We therefore shortly go into a little more detail on the most common discrete topics and how they may support mathematical competencies.

4 Potentil benefits of discrete mathematics topics for mathematics education

We see the potential benefits of teaching discrete mathematical topics in three broad areas, and some of these benefits have been highlighted in existing literature. The first potential benefit is offered by the content, which is accessible and offers interesting and relevant *topics* for teaching and learning (Anderson et al., 2004; DeBellis & Rosenstein, 2004).

The second potential benefit is the learning of mathematics and the acquisition of general mathematical *competencies* (Coenen et al., 2018; Vorhölter et al., 2019) including *affect* (Goldin, 2018) that influences the learning of mathematics and the third potential is the relevance of discrete mathematics for living in the modern world (Hart & Martin, 2018; Rosenstein, 2007). In this section, we will discuss general benefits and in Sect. 6, the benefits resulting from specific discrete mathematics topics.

Accessible topics for teaching and learning

As early as the end of the 1980s, there were calls to integrate discrete mathematics into teaching, not only in higher education but also in schools (Dolgos, 1990). Advocates of discrete mathematics have noted that problems in discrete mathematics are relatively accessible, in the sense that a student may be able to understand what a problem is asking or can explore a situation without needing a lot of prior mathematical experience (Anderson et al., 2004; Devaney, 2018; Ferrarello & Mammana, 2018; Rosenstein, 2007). This is often because the problems themselves do not require knowledge of technical definitions or specific mathematical knowledge, and students can exemplify and explore objects. The discrete nature of the objects would seem to lend itself to this accessibility. This accessibility of discrete mathematics is something that seems to be agreed upon by many mathematicians, mathematics educators and mathematics education researchers, and it is often used as an argument for the importance of the inclusion of discrete mathematics in curriculum (Anderson et al., 2004; Burghes, 1995; Dolgos, 1990; Hart & Martin, 2018).

While we note that this is an aspect of discrete mathematics that would benefit from more systematic study and research, this accessibility has come through in some research studies, but not as an explicit focus of the study. Some of the research on the teaching and learning of discrete mathematics with younger students highlights not only the accessibility of topics but that this accessibility can help students make sense of the current curriculum. For example, iterative problems and difference equations can help students learn algebra (Amit & Neria, 2008; Blanton & Kaput, 2005; Carraher et al., 2008; Radford, 2008; Rivera & Becker, 2008; Sandefur et al., 2018; Steele, 2008; Yeap & Kaur, 2008).

Even very young students can reason about combinatorial problems in meaningful ways (de Beer et al., 2015; Maher et al., 2011). English (1991, 1993) reports on young children's strategies as they engage with combinatorial problems. As another example, students can naturally 'invent' graph theory to solve a problem (Ferrarello & Mammana, 2018; Greefrath et al., 2022; van den Heuvel & Krabben-dam, 1991). We can only wonder how much better would the students' work be if they already knew some graph

theory or had previous experience with counting or recursive problems?

For many topics in discrete mathematics, students ranging from young children to undergraduate students can be posed similar questions and have a reasonable chance at investigating the problem at their different levels. This results in self-differentiating tasks that allow individual approaches to the problem (Ostkirchen & Greefrath, 2022). For example, how many 2-color towers can I make of height 5, can be extended to more complicated problems for older students by increasing the number of colors and the height of the tower. Maher et al. (2011) describe the use of combinatorial problems in such contexts among students in a longitudinal study. Students as young as the third grade can investigate recursive structures they build with toothpicks and stickers while high school students can develop recursive models of bacteria growth and the spread of epidemics, which is a simplified version of models studied by epidemiologists (Radford, 2008; Sandefur & Manaster, 2022; Yeap & Kaur, 2008). This highlights what we mean by accessibility – students can have access to mathematical topics and ideas regardless of background and prerequisite knowledge (DeBellis & Rosenstein, 2004; Ferrarello & Mammana, 2018).

Although we have several examples of empirical research that implicitly demonstrates the accessibility of discrete mathematics and the value of this accessibility, we note that there is also more work to be done. There is a need for the field to focus on the issue of accessibility more systematically and explicitly. While there seems to be some agreement on the accessible nature of these discrete problems, how accessible are they? And what do we mean by that? How is that measured? What does it mean for discrete mathematics to be accessible, and how can we tell if discrete mathematics is any more or less accessible than other mathematical tasks? We see potential value in investigating issues of accessibility in three ways.

- First, we think it is important for researchers to conduct empirical studies that explore the accessibility of discrete mathematics – perhaps through reports on mathematicians and students' views and experiences of discrete mathematics, through data that shows students without much prerequisite knowledge engaging in discrete mathematics tasks, or through explicit investigations into how students engage with both discrete and continuous concepts.
- Second, there is a need for theoretical explorations into the nature of accessibility in mathematics, exploring what makes problems or topics accessible and what might make discrete mathematics particularly accessible for students.
- Third, it's more than just accessibility that's important, it's about how it can then be used and understood. Does

For example, graphs can be considered as modeling tools

it allow for deeper understanding of mathematics (conceptual and practices)? Can we leverage accessibility to develop both deeper understanding of concepts and methods, as well as developing mathematical practices? Is accessibility a way of dealing with diverse learning groups and designing inclusive mathematics teaching?

We will go into the discussion of accessibility in more detail when we consider specific discrete topics in Sect. 6.

Promotion of mathematical processes and affect

Discrete mathematics offers opportunities for students to engage with affect in mathematics and on general mathematical processes and competencies, mentioned earlier. Even though discrete mathematics seems particularly accessible to students, this does not mean that it is easy or does not provide access to rich and powerful mathematical thinking. Indeed, while some topics in discrete mathematics are accessible in that they have easy access to engagement, they are in some cases notoriously difficult. Student difficulties are reported in many areas of discrete mathematics, e.g. in successfully solving combinatorics problems, and implementing graph theory algorithms (Annin & Lai, 2010; Batanero et al., 1997; Kavousian, 2010; Medová et al., 2019). While recursion can support students' understanding of algebra, finding explicit solutions to many difference equations is difficult or impossible (Sandefur et al., 2018). So, we argue, discrete mathematics offers an opportunity for students to think critically about important mathematical concepts and to develop competencies (Doorman & Gravemeijer, 2009).

With regard to mathematical thinking, there is ample evidence that students who engage with discrete mathematics are reasoning robustly about difficult mathematical concepts (Batanero et al., 1997; Lockwood et al., 2021; Maher et al., 2011; Weigand, 2014). In this way, discrete examples could also be used to counter the difficulties in understanding non-linear multiplicative problems (Tillema & Gatzka, 2016). Greefrath et al. (2022) deal with students thinking about real connections using graph theory, which the students came up with on their own. So, engaging with this context can promote modeling competence (Greefrath & Vorhölter, 2016).

For many discrete mathematics topics, the accessible nature of the tasks and objects lend themselves to explorations, where students can generate examples, make conjectures, observe patterns (Vogel, 2005), articulate and justify generalizations, reason with multiple representations, and engage in modeling (Doorman & Gravemeijer, 2009; Guerrero-Ortiz et al., 2018; Street & Street, 1998). In this area, discrete mathematics content can also be considered as a tool for mathematical work from a process-related perspective.

(Greubel et al., 2020; Thomas et al., 2015). Colipan and Liendo (2022), and Gravier and Ouvrier-Bufferet (2022) deal with students developing problem solving and reasoning skills through discrete puzzles. As a consequence, we can say there is an indication that topics in discrete mathematics can develop some critical reasoning and ways of thinking. In addition, we can highlight other discrete-specific ways of thinking. For example, we see opportunities for students to engage with modeling, problem solving and proof (Dawkins & Roh, 2022; Grenier & Payan, 1999).

Returning to the example of How many 2-color towers can I make of height 5 (Maher et al., 2011), we note the potential for students at a variety of ages and backgrounds to acquire mathematical competencies based on these explorations. For younger students, exploring this task may involve various processes, namely (a) selecting a desirable representation, (b) engaging in systematic, organizational listing that requires precision, and (c) justifying that they have all possible towers. For older students, they may additionally start to conjecture and generalize, articulating the number of towers that could be made of greater height or involving more colors. Indeed, Maher et al. (2011) describe students' development from building towers to connecting them to Pascal's triangle and the binomial theorem.

Motivation, emotions and beliefs as components of affect can have an influence on the acquisition of mathematical competencies (Di Martino & Zan, 2010; Hannula, 2012). Topics in discrete mathematics are seen as suitable for conveying a new image of mathematics to teachers and students. Mathematical discoveries on problems that are not part of the teaching routine are easier to make here than in many other areas of mathematics. This is true even for students who may be considered less able (Goldin, 2004). For example, is game theory suitable for promoting affective, meta-affective, and conative affordances? To learn this, various aspects must be taken into account so that this potential of discrete mathematics can be exploited. A potential problem is not explicitly addressing affect and relying on the fact that discrete mathematics is motivating to keep students engaged. Interest may not be enough to keep them working. A second problem can arise if one is too quick to introduce the conventional terms and solution procedures in different areas, such as graph theory and difference equations, without giving students the opportunity to discover things for themselves (Goldin, 2018).

Relevance of discrete mathematics

The relevance of discrete mathematics can be seen broadly in the "key themes" highlighted in the curricular

characterization of discrete mathematics that we proposed in Sect. 2. Key themes addressed through discrete mathematics include: networks, enumeration, sequential change, strategic decision

making, fairness, and the Internet. Networks (addressed by graph theory) are ubiquitous in modern life, from transportation networks to social networks to the Internet. Enumeration (combinatorics) is concerned with finding “how many,” for example how many ways committees can be formed from given group members or how many sales routes are possible. Sequential change (modeled with iteration and recursion) is step-by-step change, like yearly population growth or hourly medicine concentration in a patient’s system. Strategic decision making (as modeled by game theory) is needed in “games” of politics, economics, or entertainment. Fairness (as analyzed through the mathematics of fair voting and fair division) is essential in democratic societies to ensure fair elections, fair representation, and fair allocation of goods and services.

Another content aspect that is relevant to mathematics education has to do with equity and social justice (D’Ambrosio, 1999). Mathematics education should also support students’ better understanding of social, political and justice issues in our world. Addressing social issues is very possible within the framework of mathematical modeling. It is recommended that more emphasis be placed on mathematical modeling, which opens up the possibility of addressing social issues in school mathematics (Julie & Mudaly, 2007). Many of these models use discrete mathematics as a basis.

The digitization of society in the twenty-first century requires a focus on mathematical competencies that complement the work of computers. An important element of these competencies concerns the understanding of the mathematics underlying the mathematical work that computers do (Gravemeijer et al., 2017). Understanding how computers work and their applications requires knowledge of discrete mathematics (Pollak, 2007). However, the special opportunities provided by technology are also pointed out in the context of teaching discrete mathematics (Durgeva & Varbanova, 2017; Weigand, 2014). In the context of a short Canadian experiment to explore the potential of introducing complex dynamical systems into curricula, it was discussed that the transition to a computer treatment involves as a fundamental element one or more types of discretization, for example of time, of space or of matter. Overall, computer simulation played an important role here (Caron, 2019). Ziegenbalg (1984) sees discrete models in the context of computer use in mathematics education.

Because of the fact that computers require discretized information and all numbers are represented using bits and are approximations of real numbers, the computer necessitates discrete mathematics. Thus, it is not an exaggeration to say that discrete mathematics is a key feature for preparing students to be well-versed in computers and in programming. In this way, discrete mathematics topics are related

closely to the kinds of things that are necessary for computer

science—combinatorics, logic, recursion, iteration, all emerge as relevant topics with clear applications in computer science. This is one way in which discrete mathematics truly is the mathematics of our modern world. Because we want students to use computers and to harness their power, we literally need them to have knowledge of discrete mathematics to do so effectively. In connection with the work with computer algebra systems, difficulties could also be found due to the limitations of the technology through discrete structures and finite precision in connection with the representation of functions (Artigue, 2002; Zbiek et al., 2007). Here, a conscious look at discrete structures offers the chance to better understand the outputs of the technology.

Recently, some researchers have also examined more explicitly connections to computing, looking at ways in which programming may support and enrich students' combinatorial reasoning, and, conversely, how combinatorics may be a useful topic in introducing programming. Findings suggest that there is promise in leveraging natural aspects of programming (iteration, nested loops) in supporting ideas like the multiplication principle in counting (Lockwood & De Chenne, 2020, 2021). Because computers must deal with discretized data, there are more ideas within discrete mathematics that can be examined in the context of computing; such ideas are ripe for more investigation.

modalities; visual, auditory, tactile, kinesthetic. Some students are solitary learners while others are social learners. As a person learns, there is an interaction between what is being taught and the student, which differs from student to student. For example,

5 Relation between discrete mathematics and students' cultural upbringing and learning style

Some students do well with the algebra-to-calculus curriculum sequence, but others struggle. One problem may be that the language of algebra itself can be puzzling to a novice. The variety of ways that variables are used causes confusion for some students: as a changing value; a known quantity; two quantities, one whose value is treated as known and the other value which must be found (Moss & Lamberg, 2019). For example, colleagues have reported that some of their college students think that $f(x)$ means multiply f by x . The confusion with variables may be less of a problem with many of the topics of discrete mathematics. Vertex-edge graphs are more about reasoning than about variables, and there is a visual component connected to the reasoning. Iterative models can be analyzed using spreadsheets which help clarify the meaning of the variables and their relationships. There may also be a conflict between the way mathematics is taught and the way the student learns. Students gain knowledge through different sensory

Shade (1997, p. 83) notes that 'a pictorial image should not be expected to convey the same information to all individuals or groups.' While the way students learn is clearly individual, there is considerable evidence that students raised in different cultures have different learning styles. In a comparison of students from the USA, Hong Kong, Japan and Korea, Ma and Ma (2014) found that cooperative learning methods had a greater effect on the mathematical performance of the East Asian students than these methods produced in the United States. On the other hand, in an exploratory study of South African mathematics students, Bosman and Schulze (2018) found that different learning styles were conducive to better or worse performance. In this study, the higher achievers tended to prefer to work individually. The authors recommended further exploratory research be conducted to determine the impact of demographic variables on learning style and achievement. We note that many discrete mathematics topics, including solving discrete puzzles or playing then analyzing simple zero-sum games (discussed later) lend themselves naturally to cooperative learning settings, thus possibly appealing to students who learn better in a cooperative learning setting. As these studies indicate, cooperative learning is not necessarily better for weaker students or for stronger students: it depends on the student and possibly their culture.

A study of the learning styles and study strategies of Brunei secondary students (Shahrill et al., 2013) found that the high mathematics achievers have a more auditory-language learning style than the less successful mathematics students. They also found that females who were more successful in mathematics scored higher on visual-language and auditory-visual kinesthetic learning styles than their male counterparts. They argued that it might be effective to coach low achieving students in the use of learning styles and study strategies that empirical research identifies as possibly helpful. A Russian study (Sheromova et al., 2020) surveying a variety of literature, discussed the different learning styles of students, some of which may be based on left-brain versus right-brain dominance. As such, they summarized different teaching methods that might work for different students, depending on their learning style. Algebra and calculus have traditionally been taught using only one approach, although increasingly, algebra and calculus are being taught using active learning methods. Discrete mathematics offers a variety of topics that can naturally be taught in a variety of ways, including auditory-visual and visual-language.

To further support this point of view, Nasir et al. (2008) includes a nice collection of studies on the relationship between learning mathematics, culture, and learning styles. A collection of studies reports on how African American high school basketball players, Brazilian street vendors, Liberian Kpelle students (primarily rice farmers), and oth-

made sense of mathematics through relevant contextual situations. This agrees with Shade (1997) who stated that research indicates that African Americans students gain their knowledge 'most effectively through kinetic and tactile senses, through the keen observation of the human scene, and through verbal descriptions.' Other articles (Berry, 2003; Boykin, 1986; Dance et al., 2000; Stiff, 1990) also suggest dealing with the different learning styles of African American students could result in an improvement in their mathematics understanding and learning. These studies indicate that some students, especially social learners, may greatly benefit from discrete mathematical modeling of real-world contexts. These results indicate that students such as these might respond better to mathematics introduced by recursive topics, such as population growth, buildup and elimination of chemicals in the body, the spread of epidemics (Sandefur & Manaster, 2022), to name a few. These recursive models can actually support the learning of algebra, as is discussed later. Or maybe students respond to the visual nature of graphs related to topics such as the traveling salesman problem. Gravier and Ouvrier-Buffet (2022) describe a visual puzzle whose analysis leads to an understanding of the difference between local and global maximum and minimum. If mathematics provides a means to study a problem and analyze a means to improve the situation, these students have a reason to put in the effort to learn the mathematics.

We note that Shade (1997) indicated that there is insufficient evidence to determine what changes are necessary to improve the learning of African American students, and this is clearly true of the other students described in the previously mentioned studies. This indicates a need for a cadre of researchers to determine how to effectively address the needs of students with different learning styles. As stated in NCTM (2000, p. 14), 'Teachers need help to understand the strengths and needs of students who come from diverse linguistic and cultural backgrounds, who have specific disabilities, or who possess a special talent and interest in mathematics. They can then design experiences and lessons that respond to, and build on, this knowledge.'

As this research indicates, students who come from different cultures, are of different genders, and/or who have different learning styles may perceive mathematics in different ways. In previous sections, we discussed three aspects of discrete mathematics: accessibility, promotion of mathematical processes, and relevance. The questions that need to be answered are, how, in what ways, and for what students might discrete mathematics, integrated appropriately into the curriculum, provide an improved access into mathematical processes and competencies? For different students, the answers will surely be different.

Discrete mathematics is accessible and can be visualized because of the discreteness. Through drawing graphs or constructing discrete objects

with available manipulatives, discrete mathematics can be both tactile and kinesthetic in ways that currently taught material in the early years often is, and in late secondary school is not. The problems lend themselves to cooperative learning for the social learner. Many of the problems can be made to relate to the students' environment, and thus provide meaning and relevance of mathematics to real problems in the students' world. It is important to note that we are not proposing the teaching of discrete mathematics because weak students cannot handle algebra. We are proposing that discrete mathematics might provide a better access to mathematics for some students because it incorporates our ideals of mathematical thinking within a context that may appeal to different learning styles or different cultural underpinnings. In addition, as discussed elsewhere, many discrete topics support the learning of traditional curricula. Might this alternate approach to the curriculum work for students for which the current approach does not? This is an area that should be studied.

6 Description of discrete topics included in papers in this special issue

A number of authors advocate the inclusion of discrete mathematics in the curriculum. Some discrete topics support and extend the more traditional curricula of various countries while other discrete topics enable a range of topics that do not fall into the traditional algebra to calculus paradigm (Anderson et al., 2004; Burghes, 1995; Dolgos, 1990; Hart & Martin, 2018).

We now describe the discrete topics for school mathematics discussed in other papers in this issue: combinatorics, number theory, graph theory, recursion, and discrete games and puzzles. In doing so, we characterize them in terms of some of their compelling pedagogical features, highlighting how some might enhance and support current curriculum while others may extend it. We also describe some of the relevant research on the teaching and learning of each of these topics.

We intend that this deeper discussion of these topics will lead to a stronger consideration of them in how they support and enhance both the learning of already established curricula, and the development of desired mathematical competencies. Through a stronger interconnection of topics, to which the study of integrated areas of discrete mathematics can contribute, relationships to the real world can also become clearer (Street & Street, 1998). Discrete mathematical topics are thus an integral part of mathematics, and they help broaden and modernize the school curriculum, supporting desired mathematical processes and competencies (Anderson et al., 2004; van Drunen, 2017).

Number theory

We start our discussion with number theory, because it is the discrete topic most people are familiar with. Number theory is primarily considered the study of properties of the integers. The integers represent a discrete structure, and thus the ways in which students may engage with number theory suggests they are engaging with a discrete topic. Throughout the grades, students study properties of integers, such as

- multiplication and division algorithms,
- number systems and bases,
- prime numbers and the prime factorization of natural numbers, and
- Pythagorean triples.

These topics make number theory the discrete topic that is perhaps the most pervasive in the schools, particularly in the lower to middle grades (Anderson et al., 2004).

The competency of proving and reasoning has received considerable attention in the research community (Gardiner, 2004; Heinze et al., 2004). As suggested by Dawkins and Roh (2022), number theory presents a fertile area for understanding the ways students reason. In this paper, college students compare arguments about statements such as

- If integer x is divisible by 2 and by 7, then x is divisible by 14
- If integer x is divisible by 4 and by 6, then x is divisible by 24

Number theory topics can contribute to students' problem solving, and reasoning and proof abilities (Anderson et al., 2004) through problems related to greatest common divisor, modular arithmetic, mathematical induction, check digits, number systems, and others. While these topics are generally included in curricula, more research is needed to investigate effective ways to teach number theory topics to students, as well as to explore how these topics support, and are supported by, students' problem solving, reasoning, and proof activities.

Integers are pervasive in the lower to middle grades. Understanding the integers within the base 10 system and being able to convert numbers to other bases and operate with them can deepen students' understanding of standard arithmetic algorithms they learn with integers. Other bases can also serve students in their understandings of logarithms and exponents, and in working with binary digits in computing. Further, integers form the core for young children to begin mathematical modeling because they can both visualize numbers of objects through pictures and

can manipulate them with the use of blocks or tiles.

For

example, Colipan and Liendo (2022) consider the problem of how many squares can one square be partitioned into. Students can draw squares within squares to investigate this problem. It is clear that for any integer n , a square can be partitioned into n^2 squares. What other integers work and don't work? Problems such as this develop problem solving skills, modeling and the use of a variety of different representations.

In a very real sense, integers and natural numbers are the building blocks for the mathematics that comes later.

Combinatorics and enumeration

Combinatorics is a domain that deals primarily with enumeration. Like number theory, combinatorial enumeration is related to natural numbers, but here the focus is on determining cardinalities of sets of outcomes and ways in which to enumerate discrete objects. Most practically in school mathematics, it involves the solving of "counting problems," in which we determine how many outcomes satisfy certain constraints. Such problems are important not only for practical applications (such as determining the strength of a password system), but also because they tend to be accessible problems that require little mathematical background but can still be challenging to master. Further, such problems lend themselves to desirable practices such as generalizing (Lockwood & Reed, 2021; Reed & Lockwood, 2021), proving, using multiple representations, and attending to precision (Lockwood et al., 2020a, 2020b; Maher & Martino, 1996; Maher & Speiser, 1997). Combinatorics ranges from simple counting problems involving Cartesian products (English, 1991, 1993) to more sophisticated counting situations that require clever ways to encode outcomes (Lockwood et al., 2018), and it includes topics like the binomial theorem and combinatorial proofs of binomial identities. As such, it hits upon relevant aspects of the school curriculum from the primary grades through the secondary grades, and into college. In addition, these problems lead to important connections to probability, statistics, and computer science. Lockwood et al., (2020a, 2020b) provide a comprehensive review of research on combinatorics education.

Combinatorics has received considerable attention in the mathematics education research literature within the past few decades (Batanero et al., 1997; Coenen et al., 2018; English, 1991; Hurdle et al., 2016; Lockwood, 2013, 2014, 2022; Schuster, 2004; Tillema, 2013). It is also included, to some degree, in the curricula of many countries including Israel, Brazil, Germany, Spain, and Hungary; because of its role in the curricula, considerable work has been done in investigating the teaching and learning of combinatorial topics among students at a variety of ages in these and other countries (Batanero et al., 2005; Borba et al., 2011; Hart &

There is ample evidence that counting can be difficult for students, particularly because counting problems can appear to lack reliable solution procedures and problems that are similar in structure can appear different from each other (Annin & Lai, 2010; Batanero et al., 1997). Researchers have noted that students can find it difficult to understand and justify counting formulas and distinguish between problem types, and that it is difficult to verify counting problems (Batanero et al., 1997; Eizenberg & Zaslavsky, 2004; Lockwood et al., 2015). In light of such challenges, though, many researchers have attempted to identify ways to improve student success in solving counting problems. This has included identifying productive ways of thinking, identifying potentially useful instructional interventions, and examining ways in which students think about particular ideas and topics in combinatorics, such as Cartesian product problems, multiplication, equivalence, binomial coefficients, combinatorial proof, and symbolization and representations (Coenen et al., 2018; English, 1991; Erickson & Lockwood, 2021a; Halani, 2012; Lockwood, 2014; Lockwood & Purdy, 2020; Lockwood & Reed, 2020; Lockwood et al., 2021; Maher & Speiser, 1997; Maher et al., 2011; Montenegro et al., 2021; Soto et al., 2022; Tillema, 2013, 2014, 2018; Tillema & Gatzert, 2016; Wasserman & Galarza, 2019).

In addition to studies that explore combinatorial topics in and of themselves, there has also been work that has examined other practices and competencies in the context of combinatorics. This has included work that has examined generalization, computing, problem solving, and proving in combinatorics. Such work has emphasized ways in which combinatorics can support students' practices and competencies (Coenen et al., 2018; Ellis et al., 2021; Erickson & Lockwood, 2021a, 2021b; Lockwood, 2022; Lockwood & De Chénne, 2020, 2021; Lockwood et al., 2020a, 2020b; Maher & Martino, 1996; Maher et al., 2011; Reed & Lockwood, 2021).

In this issue, several papers focus on the teaching and learning of combinatorics. These include;

- examining ways in which combinatorial tasks can support teachers' algebraic reasoning (Tillema & Burch, 2022),
- focusing on productive ways of thinking about combinatorics in a high school discrete mathematics setting in the United States (Soto et al., 2022),
- examining ways in which a middle and high school student used shifts in representational registers to reason about a formula for binomial coefficients (Lockwood & Ellis, 2022), and
- developing frameworks for students' combinatorial reasoning (Antonides & Battista, 2022).

Such work will serve to further the field, and it represents the many different approaches and perspectives that researchers are currently taking to improve the teaching and learning of combinatorics as a key aspect of discrete mathematics. Because of combinatorics' natural connections to domains like probability, statistics, computer science, research on its teaching and learning will likely evolve as the field's understanding of those topics evolve as well. In this way, investigations into teaching and learning combinatorics can flexibly adapt to include new kinds of questions and directions.

Iteration and recursion

By iteration, we mean the repeated application of a procedure to the previous application of the procedure. This topic is discrete in a different sense from number theory and combinatorics, two topics that deal specifically with integers. Iteration is discrete in the sense that there are an integer number of steps, but the resulting numbers often come from the real numbers, not the integers. In this article, we use iteration and recursion interchangeably, although in computer science, recursion usually means something different.

Recursive thinking has been used for thousands of years. Archimedes approximated pi by finding the circumference of polygons with an increasing number of sides. Fibonacci discussed the second order difference equation $a_{n+2} = a_{n+1} + a_n$ in the thirteenth century. Pascal introduced his triangle in the seventeenth century. The recursive proof technique of mathematical induction has been used informally for over two thousand years, and its modern form has been used for several hundred years, in one form or another. Before the development of calculus, much mathematics was done using iteration. The development of calculus made the need for difference equations less important, and thus the teaching of iteration in the K-12 curriculum became almost non-existent. The pervasiveness of computers, particularly spreadsheets, has swung the pendulum back to where iteration is quite important again, as discussed below.

There is research that iterative problems support students' learning of algebra. For example, as seen in Fig. 1, Rivera and Becker (2008) had students construct a line of n squares out of toothpicks, which they did by repeatedly

adding three toothpicks at each stage. From this, students generated the formula $3n + 1$ for the number of toothpicks used to construct n squares. The students went from the recursive pattern of adding three, which we could write using the difference equation,

$$s_{n+1} = s_n + 3,$$

to the explicit algebraic formula.

Amit and Neria (2008) had students considering a Hanukah problem in which on each day, one more candle was lit than the previous day. From this problem, which can be written as

$$s_{n+1} = s_n + n + 1,$$

students were able to generate the quadratic formula for the total number of candles lit.

Similar results showing how iterative problems both engage students and support the learning of algebra have been obtained by others (Blanton & Kaput, 2005; Carraher et al., 2008; Radford, 2008; Steele, 2008; Yeap & Kaur, 2008).

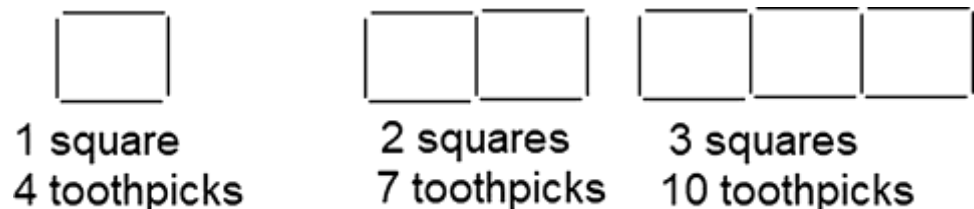
As described in these papers, recursive problems with simple constant change, such as the add 3 toothpicks problem, or linear change, light one more candle than the previous day, have been shown to support younger students' learning of algebra by generating meaningful formulas from the context of the situation. Given this, it is surprising that there has been little research on the classical and well-known topics of growth and decay processes and how they can support the learning of algebra. These problems occur when there is constant proportional change: the addition or subtraction of the same proportion at each stage. This occurs in the addition of interest, simple population growth, the removal of a fraction of radioactive material, or the removal of a certain proportion of medicine from the body. Such problems result in difference equations of the form

$$s_{n+1} - s_n = \pm ps_n \text{ or } s_{n+1} = (1 \pm p)s_n.$$

Similarly to the repeated addition or subtraction, such equations are just repeated multiplication which results in an exponential function,

$$s_k = (1 \pm p)^k s_0.$$

Fig. 1 Recursive procedure for constructing a line of squares



These authentic and interesting examples from the real world of the students unite iterative processes with an understanding of exponential functions, just as the constant addition problems develop students' understanding of linear functions (Castillo-Garsow, 2013; Schonger & Sele, 2021).

Discrete recursive models allow calculations with a spreadsheet (Keune & Henning, 2003). This easy access also allows the comparison of different approaches and gives insight into the influence of the assumptions made at the beginning of the modeling process. This approach therefore supports a deep understanding of characteristics of one of the competencies of mathematical modeling. These ideas have already been examined in empirical studies: For example in a small case study on mathematical modeling of growth and decay processes with exponential functions, it could be shown that "thinking continuously is fundamentally different from thinking discretely" (Castillo-Garsow, 2013, p. 1451). A further study reported that students' understanding of discrete versus continuous graphs of growth processes, which result by taking the limit as h goes to 0, might also be influenced by the nature and representation of underlying variables (Leinhardt et al., 1990). This topic area is interesting because the problems can be modeled

both continuously and discretely (Guerrero-Ortiz et al., 2018). Studies from the field of modeling also show that the consideration of a change can be modeled continuously or discretely, and that the discrete view is an important extension (de Beer et al., 2015; Doorman & Gravemeijer, 2009; Guerrero-Ortiz et al., 2018).

As mentioned in Sect. 5, it is possible that recursive models can support some students' engagement with topics of social importance, such as drug models, epidemics (Sandefur & Manaster, 2022), population growth, change in genetic makeup, or management of renewable resources. It could also be that recursive models, similar to the toothpick construction, can engage students just by being fun.

We note that there are connections between recursion and combinatorics in that some counting problems can be solved recursively, for example, the counting of the number of Hanukkah candles lit (Amit & Neria, 2008) and the number of seats in a theater (Burrell et al., 1991). In fact, the binomial coefficients can be found recursively using Pascal's triangle.

We suggest that recursive problems support students' mathematical competencies by having them reason about variables and functions through meaningful contextual situations. For more discussion on the possible connection between recursion and learning algebra, see Sandefur and Manaster (2022). There are several additional papers which suggest a variety of ways that the introduction of recursive problems can possibly support, not only the learning of algebra, but the development of a variety of mathematical processes and competencies (Doorman & Gravemeijer, 2009; Rosenstein, 2007; Sandefur et al., 2018).

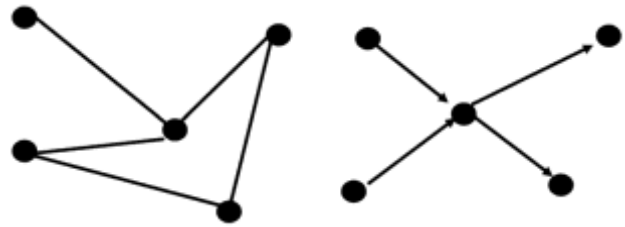


Fig. 2 Vertex-edge graph on left, directed graph on right

For several decades, some mathematicians and mathematics educators have promoted the inclusion of iteration and recursion throughout the school curriculum, but it has not happened. Our hope is that new research studies in mathematics education will show that the introduction of recursive problems, combined with the use of spreadsheets, (a) supports a deeper learning of many algebraic concepts, (b) improves the understanding of the different uses of variables, (c) increases students' appreciation of the usefulness of mathematics, and (d) improves students problem-solving skills. Armed with evidence of the advantages of teaching recursion, we will then be able to make an even stronger case for its inclusion as a core part of the curriculum.

Graph theory

By 'graph theory', we mean vertex-edge graphs, examples of which would be flowcharts and organizational charts. Figure 2 displays a (vertex-edge) graph on the left, in which vertices are connected by edges. On the right in Fig. 2 is a directed graph in which the edges have direction. A street map in which the streets are all two-way could be considered as a graph where the streets are the edges, and corners where streets meet are the vertices. A street map in which the streets are all one-way would be considered a directed graph. One of the first applications of graph theory is Euler's Königsberg bridge problem in which the edges are bridges and the vertices are the different land-masses connected by the bridges (Grötschel & Yuan, 2012).

Vertex-edge graphs provide discrete mathematical models for many situations involving networks or relationships among a finite number of objects. Examples include:

- Solving problems related to prerequisite relationships (e.g., project scheduling with *critical paths* and *PERT charts*)
- Solving problems related to conflict relationships (e.g., managing conflicts in chemical storage, radio interference, or social conflict through *vertex coloring*)
- Finding optimal routes and networks (e.g., shortest travel paths, least expensive street maintenance routes, optimal sales networks, data structure trees, and models for social

networks and the Internet, using graph theory topics such as *shortest path, Euler paths, Hamilton paths, trees, minimal spanning trees,*)

We note that several discrete topics can be integrated together to support understanding. Suppose students are given the combinatorial problem to count the number of handshakes if everyone in a group of n individuals shake hands. This problem can be visualized as a vertex-edge graph where the vertices are the people, and an edge is a handshake between two people. This problem can also be studied recursively by looking at the number of additional handshakes if another person is added, which gives the difference equation

$$s_{n+1} = s_n + n,$$

which has the explicit (n choose two) combinatorial solution

$$n(n-1)/2.$$

Thus, students are considering a simple combinatorial problem both graphically, iteratively, symbolically, and algebraically. Note the similarity of this problem to the Hanukkah candle problem in Sect. 6.3.

Since Euler's development of graph theory with the Königsburg bridge problem in 1736, the study of vertex-edge graphs has been a significant field of mathematics (Grötschel & Yuan, 2012). The problems can become quite difficult, such as finding a minimal path for a traveling salesman, which is NP-complete. Other problems, such as finding Euler paths and Euler cycles are relatively easily solved using simple algorithms. Graph theory is accessible, visual, can relate to physical situations with which students can actively engage, and thus, can be introduced in elementary school (DeBellis & Rosenstein, 2004; Ferrarello & Mammana, 2018).

While many classroom implementations of graph theory repeat the same simple examples, such as the Königsberg bridge problem, again and again, without effectively developing graph theory through the grades, clear sequenced recommendations for teaching and learning graph theory are available (DeBellis et al., 2009; Hart et al., 2008). We encourage researchers and curriculum developers to consider these sequenced grade-band recommendations, as they develop curriculum materials, design research studies, and possibly propose other recommendations.

Graph theory comprises important concepts, develops thinking processes that most current curricula encourage, and thus warrants inclusion in the school curriculum. However, there are only a few studies in this area so far (Ferrarello & Mammana, 2018; Ferrarello et al., 2022; Medová et al., 2019). Greefrath et al. (2022) shows how, because of the ease of accessibility of graph theory, students can develop models related to their world and address issues of optimization.

This is very well possibly due to the given real situation if the students have no experience with graph-theoretical topics. Medová et al. (2019), compared the work of university students on several graph theory problems; the Chinese postman problem, the shortest path problem, and the minimum spanning tree problem. More research is necessary in order to identify what and how students of different ages learn when studying vertex-edge graphs, and what are effective methods for teaching and learning this topic.

Discrete games and puzzles

The popularity of both Martin Gardner's puzzle columns in Scientific American and the Rubik's Cube, among other

puzzles, show the appeal of discrete puzzles and games. They involve many of the competencies that many countries' mathematics curricula are promoting, such as reasoning, and problem solving (Stein, 1999). Discrete games and puzzles are topics not normally promoted for inclusion in the school curriculum. We believe this is a mistake and that this topic has great potential for not only engaging students, but in furthering the development of mathematical processes and competencies (Scholz, 2007). On the other hand, there does not appear to be a large amount of mathematical education research on the affordances of puzzles in students' learning.

There are discrete games and puzzle which can be used to support the learning of a variety of areas of the current curriculum. These games can also appeal to a variety of learning styles and develop several of the desired competencies mentioned earlier. We already mentioned such a discrete puzzle (Colipan & Liendo, 2022) in which students try to see how many different numbers of squares a given square can be partitioned into. This problem supports problem solving and communication while supporting the understanding of some simple geometry related to squares and rectangles. Gravier and Ouvrier-Buffet (2022) discuss the puzzle of putting some number of traps in a garden to guarantee stopping a pest. This problem involves existence, or sufficiency, in that if we put n in the garden that clearly prevents the pest, then n works. It deals with non-existence proofs in showing that no matter how n traps are set, it is possible for a pest to invade. Interestingly, this puzzle also deals with optimization in finding the minimum number of traps needed. This puzzle goes further into optimization by showing the difference between minimal, an arrangement of n traps that cannot be reduced, and minimum, the fewest number of traps needed if arranged properly, which can be lower.

Nim games involve two (or more) players who take turns removing objects from one or more piles of objects under certain conditions. One of the simplest games is when there is one pile of items and each of two players takes turns removing 1, 2 or 3 items, with the player removing the last

item winning. Suppose we are interested in how many items must be in the pile so that the player going second can ensure a win. Students, working recursively, can first discover 4 works, then 8, and, similar to the discussion in Sect. 6.3, discover the formula that if there are $4n$ items, then the player going second can always win. The argument is that whatever player one chooses, 1, 2, or 3 items, player 2 can choose the opposite, 3, 2 or 1 for a total of 4, reducing $4n$ to $4(n-1)$. The inductive thinking developed in playing this game is the basis for proof by induction. We also note that the game of Nim can be made as easy or as difficult as one wants by varying the number of piles and the number of items a player can pick up on their turn, allowing students at different levels to be both challenged and successful.

Some puzzles involve both optimization and recursion, such as the Tower of Hanoi, in which some number of disks must be moved from one peg to another in a minimum number of moves. If there is one disk, 1 move is required, two disks require 3 moves, three disks require 7 moves, and students discover that if n disks require m_n moves, then $n+1$ disks require $2m_n + 1$ moves. They also can develop the algebraic solution that $m_n = 2^n - 1$, connecting students' algebraic skills through the puzzle and recursion.

Does solving such puzzles develop students' reasoning abilities as well as their communication skills when they have to explain their solutions? Puzzles such as these may help students develop comparable skills to those used by mathematicians when faced with a research problem (Colipan, 2018): experimentation, conjecturing, building models, and construction of arguments.

We believe that mathematics education research will show that some games and puzzles, such as the ones mentioned here, have the potential to support the learning of the current curriculum, such as an understanding of optimization, the development of geometric understanding through explorations, and support for learning algebra through recursive solutions, as also discussed in Sect. 6.3. In addition, we hope it shows that student engagement supports their problem solving and communication skills. Research is also needed to (a) determine which games support what learning at what age levels, (b) which learning styles are supported by which games, and (c) which games are appropriate at which grade levels. Once there is research showing these affordances, the games and puzzles should be easy to incorporate into the curriculum as most are easy to learn.

7 Other relevant discrete topics

There have been calls by many mathematicians and mathematics educators to include a variety of other discrete topics in the mathematics curriculum, such as game theory, the

mathematics of fairness, information processing and cryptography (DeBellis & Rosenstein, 2004; Hart & Martin, 2018; NCTM, 1989; Rosenstein, 2007). Each of these topics offers something unique in student learning. The amount of research on the affordances of students learning these topics is minimal, at best.

We can only conjecture why these topics have not had a greater impact in the school curriculum. First, teachers are generally not familiar with these topics, so teacher preparation would need to be considered. Second, there has been little in the way of coherent proposals for the developmentally appropriate teaching of any of these topics throughout the curriculum, nor has there been much in the way of age-appropriate materials introduced into the textbooks for children throughout the grades. Third, many mathematicians view these topics as tangential to mathematics.

Many readers may not be familiar with these topics. Because of space issues, we give a brief overview of just two of them, including some examples that explicate some of the possible affordances of them. We hope this discussion furthers research on these topics.

Game theory

Game theory was primarily developed by the mathematician John von Neumann and economist Oskar Morgenstern in the 1940s. Game theory has become an important tool in economics and simple games help explain a variety of human behavior. The idea behind game theory is developing an optimal strategy, but it also involves some simple probability. Optimality is an important mathematical topic that is mostly ignored until algebra and calculus, in which case, only one type of optimality is considered, maximizing or minimizing functions. Because game theory involves playing simple games to get a sense of what it means to play optimally, it should work well in a cooperative learning setting. The basics of game theory are very intuitive, using some elementary algebra and making sense of simple matrices. Given its importance in a variety of disciplines, it is surprising that game theory has not made more inroads into the curriculum. Game theory can both show how to make rational decisions and can help in the understanding of seemingly contradictory behavior of people. There are several different types of simple games where students might not only develop reasoning and problem-solving skills but use some simple algebra and probability. Since game theory is a topic that many readers may not be familiar with, we give some examples and discuss how they might support the curriculum and develop reasoning abilities.

We first discuss how zero-sum games can support the current curriculum through solving simple equations, combined

with a little probability. Consider the simple two-person game as seen in the matrix

$$A = \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}$$

Player 1 picks a row and simultaneously, player 2 picks a column and the number at the intersection is how much player 2 pays player 1 (or player 1 pays player 2 if negative). In this case, player 2 will always pick the second column since its payoffs are always better, no matter what row player 1 picks. Similarly, player 1 will always pick the first row. So both players break even. This is an example of a pure strategy where each player maximizes their results, a simple example of an optimization problem. Young students should be able to develop this logic through playing this game numerous times.

Mixed strategy zero-sum games involve reasoning, simple algebra, a little probability, and optimization. Consider the game given by the matrix

$$B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

under the same rules as above. In this case, there is no pure strategy: if player 2 always chooses column 1, then player 1 will always choose row 2, but if player 2 always chooses column 2, player 1 will always choose row 1. This is an example of a mixed strategy game; each player has to mix up their choices so the other player cannot take advantage of them. Suppose the probability player 1 chooses row 1 is p . Then if player 2 chooses column 1, the expected payoff for player 1 is

$$(-1)(p) + (0)(1 - p)$$

while if player 2 chooses column 2, the payoff for player 1 is

$$(1)(p) + (-1)(1 - p)$$

Player 1 should choose a strategy such that the payoff is the same no matter what player 2 chooses, so

$$(-1)(p) + (0)(1 - p) = (1)(p) + (-1)(1 - p).$$

The solution is $p = \frac{1}{3}$, so player 1 should choose row 1, one-third of the time, and row 2, two-thirds of the time, resulting in an average payoff of $-\frac{1}{3}$. A similar calculation results in player 2 choosing column 1, two-thirds of the time and column 2, one-third of the time. Mixed strategy games occur quite frequently in life and are important in understanding economic behavior.

There are a number of other game theoretic situations that might deserve exploration, such as non-zero-sum games like the prisoners' dilemma, an example seen in the matrix

$$C = \begin{pmatrix} (1, 1) & (-4, 3) \\ (3, -3) & (-1, -1) \end{pmatrix}.$$

The first number is the payoff to player 1 and the second number is the payoff to player 2. In this matrix, player 1 selects a row and player 2 selects a column. Note that whatever row is chosen by player 1, player 2 gets a better payoff by choosing the second column. Whatever column player 2 selects, player 1 gets a better payoff by choosing the second row. This is a pure strategy game in which both players choose the second option, resulting in their both losing 1. If they had cooperated, they could have both won 1. Playing in their own self-interest results in them both losing instead of winning. This is again an optimization problem in which if the players cooperate, they can achieve their maximum payoff, but if each plays in their own self-interest, they achieve a smaller payoff: This is similar to the difference between minimum and minimal, discussed in Gravier and Ouvrier-Bufferet (2022).

Some games involve simple algebra and probability. The reasoning involved in games relates to several mathematical competencies: problem solving, modeling, optimization and communication, all within an engaging social context. Game theory seems to be a discrete topic that is open to investigation on its affordances with students learning mathematics. There are a variety of other game situations that could be considered (Scholz, 2007; Stebler et al., 2013).

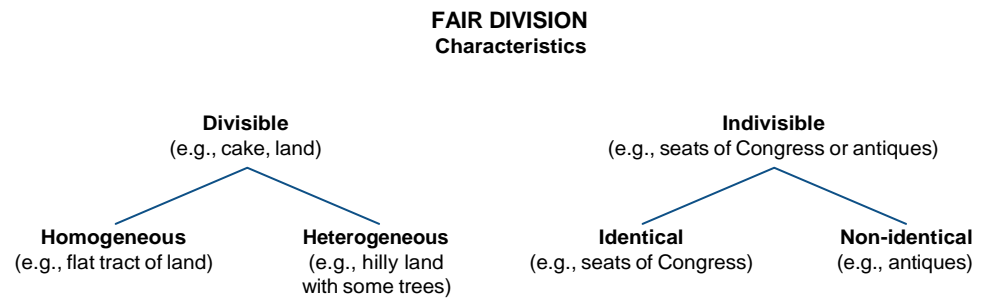
Mathematics of fairness

Fairness is an important issue in students' lives and in society and government. Some of the mathematics of fairness is essential quantitative literacy for informed citizens in democratic societies. As such, fair division and fair voting are relevant topics for the school curriculum. In the following two subsections we briefly discuss some of the relevant mathematics of fair division and fair voting.

Mathematics of fair division

An aspect of fairness that students encounter in their daily lives is fair division. The essential question is: What is a fair share? One way to categorize fair division problems is in terms of the objects to be divided. First categorize the objects as *divisible* or *indivisible*. Then consider the "sameness" of the objects, that is, are they identical or not, heterogeneous or homogeneous. For example, a cake is divisible while the individual seats of Congress or Parliament or antiques in an estate are indivisible. Divisible items are sometimes called "continuous," and indivisible items are called "discrete." In the case of divisible objects, like a cake, the most interesting and challenging problems are when the divisible object is *heterogeneous*, like a cake with swirls of

Fig. 3 Categorizing fair division problems



frosting or a piece of land with hills and trees. In the case of indivisible objects, a key distinction is whether the objects are *identical*, like the seats of Parliament, or *non-identical*, like pieces of furniture in an estate. This break-down of fair division problems is shown in Fig. 3.

Division of indivisible objects is a discrete field that involves thinking with fractions and proportions and which is easily accessible at middle and secondary school levels. It should be engaging for students since it relates to issues of importance to many students, fair division of resources. It can lead to paradoxes that increase students' understanding of fractions. Since many readers may not be familiar with fair division, we present two examples of the division of identical indivisible objects. We believe problems such as these can support the current curriculum through a better understanding of fractions and the understanding of slope as a rate of change.

Suppose one student works 3 h, a second works 5 h and a third works 7 h, and there are 22 coins to divide among them for their work. Multiplying 22 by the fraction of time each student worked gives a division of

$$3 \times \frac{22}{15} = 4.40, 5 \times \frac{22}{15} = 7.33, \text{ and } 7 \times \frac{22}{15} = 10.27,$$

respectively. To give a total of 22 coins, one approach is to round the largest fraction upward, dividing the coins as 5, 7 and 10. If instead, there are 23 coins, we get the fractions 4.60, 7.67, and 10.73, so rounding the two largest fractions up has the coins divided as 4, 8, and 11, respectively. There is one additional coin, but the first student gets one less coin. How can this be fair?

Fairness examples such as this can help relate a variety of seemingly unrelated topics, proportions and lines in this case. To understand this seemingly unfair paradox, students could let x represent the number of coins. They are then evaluating the three lines

$$(3/15)x, (5/15)x, \text{ and } (7/15)x,$$

at different values of x . The slopes of the second and third lines are larger than the first, so they increase faster as x increases, resulting in their fractions overtaking the first, and the first student losing a coin.

An alternate method for dividing the coins is to round down the fractions to 4, 7 and 10 resulting in 21 coins being given. Then multiply by the smallest y that results in one of the fractions increasing to the next integer,

$$(3 \times \frac{22}{15})y = 5, (5 \times \frac{22}{15})y = 8, \text{ or } (7 \times \frac{22}{15})y = 11.$$

In this case, $y = (11 \times 15) / (7 \times 22) \approx 1.07$ is the smallest, and the third student gets the extra coin, not the first one. For 23 coins, the lowest number that results in two of the fractions equaling or being greater than the next integer is $y = (8 \times 15) / (5 \times 23) \approx 1.04$ resulting in the second and third students getting extra coins. This method takes advantage of the fact that the students who worked more hours have lines with larger slopes, and only involves solving simple linear equations.

As another example, suppose students work 13, 12 and 112 h for a total of 25 coins. In this case, after rounding up the largest fraction, the students receive 2, 2 and 21 coins. If the first student works an additional hour and the third student works 2 additional hours, the division of coins is 3, 2 and 20, by rounding up the largest fraction. So, the third student worked more additional hours and lost a coin. How is this fair? Should we instead look at the percentage of additional time worked: The first student worked $1/13 = 0.077$ or 7.7% more time while the third student worked $2/112 = 0.018$ or 1.8% more time. But should this result in the third student losing a coin? This example opens students to discussing fractions, change versus relative change, and fairness.

Fair division problems such as these only involve fractions, lines, slopes and rounding but result in what are seemingly unexpected paradoxical behavior. Could studying examples such as these, where students would come up with the different divisions naturally, lead to a different and deeper understanding of these topics, and of fairness in general? Could there be interesting student discussions about fair ways to distribute the money?

Paradoxes such as these occurred in American politics. For the first example, in 1880, it was discovered that if an additional representative was added to Congress, it would result in Alabama losing one of their representatives. In the

second example, in 1900, Virginia lost a representative to Maine even though Virginia's population grew by more than Maine's population. A third example arose when Oklahoma entered the US: Its share of representatives was added to the House of Representative, which should not have affected the other states' number of representatives, but New York lost a representative to Maine (Caulfield, 2010) (https://en.wikipedia.org/wiki/Apportionment_paradox).

Fairness problems arise with use of fractions within a context and the mathematics can relate to real world issues, which we have discussed in Sect. 5 as possibly helping social learners, among others. While fair division algorithms are intuitive, use simple proportions, are accessible to students at a variety of levels, there seems to be little research to determine if they support critical thinking and a deeper understanding of proportions and rate of change.

Another common example of fair division is fairly dividing non-identical goods, such as in an estate, among the heirs. For estate division, and other such situations that involve fair division of non-identical mostly indivisible objects, students might productively study the *Knaster's procedure* (https://mathshistory.st-andrews.ac.uk/Extras/Knaster_fair_division/), for example. In addition, each fair division method can be analyzed in terms of common fairness criteria, such as *proportionality*, *envy-freeness*, and *equitability*.

As discussed, studying fair division situations gives students valuable opportunities to develop proportional reasoning, algorithmic problem solving, and experience using mathematics to analyze abstract notions such as fairness.

Mathematics of fair voting

Fair voting is vital for democratic decision making. While there are different methods of voting, we focus here on weighted voting and ranked-choice voting, briefly summarizing these topics, highlighting some of the relevant mathematics and positing some of the educational benefits that may be achieved by including these topics in the school curriculum.

In weighted voting, each voter can cast many votes for a single candidate or choice. This may sound unfair, but in fact when it is used in certain contexts, it is perfectly reasonable. For example, consider shareholders in a corporation. Each shareholder casts a number of votes equal to the number of shares they own. The big issue in such weighted voting situations is not the so-called *weight* of a person's votes, which is the number of votes the person casts, but the *power* of their votes, which is a measure of how influential their votes are. A common method for measuring power is the Banzhaf Power Index method.

In the following simple example, we briefly describe the

where A has 3 shares (votes), B has 3 shares, and C has 1 share. Suppose a majority is required for any resolution or policy to pass. Thus 4 votes are needed for anything to pass. In this situation, nobody's votes alone have the power to pass a resolution, all possible pairs of voters have the power to pass a resolution, and all voters together can pass a resolution. Thus, in a practical sense, all voters have the same voting power, even though A and B have three times as many votes as C (three times the weight of C)! It is this power of votes that is often much more important than the number (weight) of votes. The Banzhaf Power Index gives an exact quantitative measure of this power, as follows. In this situation, there are 4 winning coalitions of which each shareholder's vote is critical in 2 of them, so the power of each shareholder can be quantified as $2/4$. In formal terms, each shareholder has Banzhaf Power Index $1/2$. Note that this method combines logical reasoning, counting the number of sets with a certain property (combinatorics), and fractions. This leads to several interesting research questions that could be explored, such as: How do such algorithmic methods of evaluating power support (a) students' logical reasoning, (b) enumeration of number of sets (discussed in Sect. 6.2), (c) their proportional reasoning, and (d) their engagement in mathematics through clearly applied settings?

In ranked-choice voting, each person ranks the candidates instead of just voting for their favorite. This gives richer data about voters' preferences, and thus allows for a result that better reflects the will of the people. Once we have collected the voting data, there are different vote-analysis methods. Common analysis methods include: (a) majority—whoever gets more than half the first-preference votes wins, (b) plurality—whoever gets the most first-preference votes win, (c) top-two runoff—use the ranked-choice voting data to immediately have a runoff between the top two first-preference vote getters (also called Instant Runoff Voting), (d) pairwise-comparison—where you see if anyone beats everyone else head-to-head (also called the Condorcet method), and (e) points-for-preferences—where you assign points for preference and whoever gets the most points wins (also called the Borda method).

For example, suppose there are three candidates in an election, A, B, and C, and voters rank the candidates according to the data in Table 1.

Table 1 Ranked choice ballots

	Rankings		
A	1	2	3
B	2	3	1
C	3	1	2
Number of Voters	16 voters	10 voters	8 voters

The plurality winner in this election is A, since A gets the most first-preference votes (16). Consider the top-two runoff method: A runs against C since they are the top-two first-preference vote getters, and B is eliminated; then the 8 voters who had B as their first preference will now vote for C since they prefer C over A; thus, C wins the runoff with A (18–16); so, C is the top-two runoff winner.

There are connections from ranked-choice voting analysis to other areas of discrete mathematics and to other mathematics. Note some immediate connections to combinatorics. You could ask students how many rankings (permutations) are possible when there are three, or in general, n , candidates; or, in the pairwise-comparison method, where all pairs are run off against each other, you could ask how many pairs are possible. Connections to graph theory are also possible. For example, the pairwise-comparison method can be modeled with a digraph (directed graph) by letting the vertices represent the candidates and two vertices are connected by a directed edge (arrow) from one to the other if the one candidate beats the other in their pairwise runoff. Then characteristics of the graph relate to the possible pairwise-comparison winner. For instance, if there is a circuit through all the vertices then there cannot be a pairwise-comparison winner. In the points-for-preferences method, students get the chance to review and practice their basic arithmetic skills as they multiply and add to find the points-for-preferences winner.

As the example above shows, different vote-analysis methods can yield different winners, and this can yield some paradoxical results. For example, the plurality winner can also be the candidate that is least preferred by a majority of voters, or eliminating a lower ranked candidate can switch the places of the other candidates. A fundamental result in this area, Arrow's Impossibility Theorem, proves that no vote-analysis method is perfect, with no paradoxical results, but some methods are better than others. Most experts agree that, unfortunately, the worst method, in terms of fairness and avoiding paradoxes, is the commonly used plurality method (Laslier, 2012).

Analyzing different voting and vote-analysis methods helps students develop their ability to mathematically model abstract real-world phenomena, like how influential votes are or how fair a voting method is. In addition, it helps develop students' logical reasoning, algorithmic problem-solving ability, communication, and general appreciation for the usefulness of mathematics. Most concretely, by analyzing fair voting, students hone their arithmetic skills and quantitative reasoning ability. We encourage more research and professional work to figure out how and where the mathematics of fair voting can be integrated into the curriculum (for example, see Hirsch et al., 2016), and how it can be effectively taught and learned. We also feel that this is a relevant and

timely topic with clear applications and potential for engagement with students at a variety of levels.

8 Looking ahead and conclusion

We have given several convincing arguments that discrete mathematics should be more strongly considered in mainstream mathematics education. These relate to its accessibility, the acquisition of general competencies, affective benefits, and the utility of knowledge of discrete topics in the modern world.

These topics are important mathematical topics that are applied widely. Graph theory is used in a variety of fields, such as chemistry, neural networks, organization. Recursion is important in computer science and iterative solutions to problems occur quite often which can lead to a better understanding of algebraic concepts: while recursion is a valuable tool which younger students seem to inherently use; it seems to disappear in later grades. Issues related to fairness and game theory are areas of intense study in political science and economics. Without exposure to these topics, we are sending out ill-prepared students.

Discrete mathematics should be incorporated into the school curriculum and educational research should be carried out accordingly. If the status of discrete mathematics remains on the fringes and not a central aspect of curricula, this poses a challenge for researchers (especially younger scholars) who are interested in studying the teaching and learning of discrete mathematics topics. It can be challenging to make a case for the importance of one's research when the topics are not themselves commonly handled in the curriculum. There is considerable research on algebra, geometry, calculus, and proof...why? These are topics that are valued, and researchers can make a case for studying them because they are what tend to get taught in classrooms. However, we argue that despite such challenges, we also need researchers to do work on topics that are not currently popular. We hope to have informed the research community about the curricular role of discrete mathematics, its educational affordances, and the current state of research in this area. We need researchers to further study how discrete mathematics is taught and learned and how it fits into mathematics classrooms worldwide. We must start somewhere. We want to encourage others to start this process so we, as a field, can better understand and appreciate the teaching and learning of discrete mathematics. Currently, the best researched topic in discrete mathematics is combinatorics, and combinatorics has found its way into the curricula of many countries for several decades. Based on the success of this growing body of research in combinatorics, we want to inspire more research in other areas. There are already

promising research efforts in the teaching and learning of graph theory and sequences (recursion).

Further, because we value discrete mathematics and its impact in society in general, we think we are doing a disservice to students who would benefit from opportunities to pursue a pathway that focuses heavily on important mathematical topics. Some mathematicians go into fields such as algebraic structures, geometry, graph theory, and a variety of other fields. We, of course, do not think less of them because they did not go into analysis: they are going into vital fields where they have strengths and interests. On the other hand, students who are not successful in algebra and calculus are viewed as (or view themselves as) poor mathematics students with no chance at being successful in mathematics. As we do for mathematicians, why don't we expose students to the variety of opportunities for rich mathematical growth and engagement in topics that are not so dependent on algebra and calculus, and even strengthen their understanding of these topics through their integration with discrete mathematics? We think that discrete mathematics offers much promise in that regard.

However, we also are interested in better understanding the teaching and learning of the different areas of discrete mathematics—not only to improve the status of discrete mathematics, but also to gain a better understanding of how best to teach each of these topics, particularly to students with a variety of learning styles. Currently, there are several topics in discrete mathematics that are simply not well understood or explored, in terms of

- how they tend to be taught,
- how students tend to reason about them (what difficulties or successes they face, etc.),
- what ways of thinking underlie them,
- how they might be used to undergird students' competencies,
- how they are different from or similar to students' reasoning about other (continuous) topics, and
- how research on each of these topics relates to existing research on continuous topics.

We contend that by engaging in research that begins to answer such questions, we can, as a field, better understand the teaching and learning of various discrete mathematics topics, contributing to a broader body of literature and knowledge. Thus, a strong motivation behind our work is to improve how discrete mathematics is taught and learned from primary to tertiary education.

We argue that improving the status of discrete mathematics as it is viewed in primary, secondary and tertiary classes is important both for the research community and for the many students who would benefit from learning topics in discrete mathematics. To summarize this section,

there are a couple of related issues to studying the teaching and learning of discrete mathematics. On the one hand, there is a need for our field to make a case for the inclusion of discrete mathematics. A different but related issue is that we simply need more research on the teaching and learning of discrete mathematics. There is a need to engage in more research for the sake of better understanding the teaching and learning of discrete mathematics, for the good of the field and the good of us as a research community.

It seems particularly relevant in the modern world to consider curricular innovation and educational research in this area of mathematics, discrete mathematics, since it is so widely used in modern applications. We hope that this paper and the following papers in this special issue will contribute to this goal and lead to more valuable work in this important area of mathematics education..

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